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Non-renormalization of next-to-extremal correlators in $\mathcal{N}=4$ SYM and the AdS/CFT correspondence

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Abstract

We show that next-to-extremal correlators of chiral primary operators in $\mathcal{N}=4$ SYM theory do not receive quantum corrections to first order in perturbation theory. Furthermore we consider next-to-extremal correlators within AdS supergravity. Here the exchange diagrams contributing to these correlators yield results of the same free-field form as obtained within field theory. This suggests that quantum corrections vanish at strong coupling as well.

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1 Introduction

Among its variety of applications, the Maldacena conjecture [1, 2, 3] has been at the origin of new results in four-dimensional quantum field theory. In particular, it has lead to the discovery of new non-renormalization properties of $\mathcal{N}=4$ Super Yang-Mills theory (SYM). Many of these properties refer to correlation functions of chiral primary operators, which are scalar composite operators in the symmetric traceless representation of the $SU(4)$ R-symmetry group with Dynkin labels $[0, l, 0]$. In [4], the AdS/CFT correspondence was used to argue that for these operators the so-called “extremal” correlation functions, in which the conformal dimension of one of the operator equals the sum of dimensions of all remaining operators, do not receive any quantum corrections. Thus, at least at large N , these correlation functions are expected to be given exactly by their free field form, which is a product of two-point functions. This conjecture has been partially confirmed for any finite N by an explicit field theory calculation to first order in perturbation theory and to leading order in the semiclassical expansion of any instanton sector [5]. The non-perturbative non-renormalization of extremal correlators has been studied by Eden *et.al.* in [6]. Using superconformal Ward identities together with the Grassmann and harmonic analyticity of the relevant superfields, these authors prove that extremal correlators can indeed be expressed as products of two-point functions. Applying the reduction formula of [7], which relates the derivative of an n -point function with respect to the coupling to an integrated $(n + 1)$ -point function with an operator insertion, they also show that extremal four-point correlators are independent of the coupling constant, under the (plausible) assumption that there are no undesired contributions from contact terms [8, 9].

Subject to the same assumption, Eden *et.al.* also prove in [6] the non-renormalization of “next-to-extremal” four-point correlation functions, for which $k_4 = \sum_{i=1}^3 k_i - 2$, with k_i the conformal dimension of each operator. In this letter we show explicitly that the radiative corrections to next-to-extremal four-point correlators of chiral primary operators vanish to order g^2 . This provides an independent check of the results in [6]. We then extend our arguments to show that, at first order in perturbation theory, next-to-extremal correlators with an arbitrary number of points are not renormalized either. We examine also the corresponding (four-point) scattering amplitudes in AdS supergravity, and argue that the exchange diagrams reduce to a free-field form. The fact that the quartic couplings have not been computed so far prevents us from giving a complete proof in AdS¹. Although the results in [6] hold for any $\mathcal{N}=2$ SCFT and for any choice of the gauge group, here we shall stick to $\mathcal{N}=4$ SYM with a $SU(N)$ gauge symmetry.

In the case of extremal n -point functions (and also of general three-point functions), there is only one $SU(4)$ invariant contraction of the “flavour” indices, *i.e.*, there is only one $SU(4)$ singlet in the product of the representations of the operators involved. This fact was used in [5] and in [11] to assign to the chiral operators a convenient irreducible representation of

¹After this work had been completed, a paper appeared in which the relevant quartic couplings are evaluated [10]. We briefly comment on the new results in the last section, and leave for future work the complete computation of the next-to-extremal four-point functions in AdS supergravity.

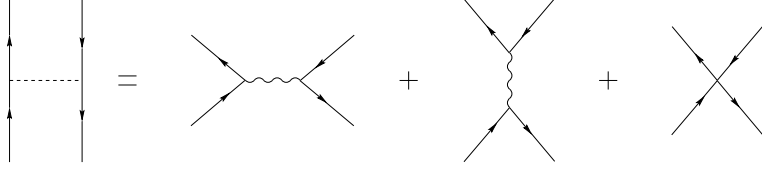


Figure 1: A dashed line connecting two solid lines indicates the effective four-scalar interaction resulting from gluon exchange and from the quartic vertex. Scalar propagators are denoted by solid lines and gluon propagators by wavy lines.

the $SU(3) \times U(1)$ subgroup of $SU(4)$ that is manifest in the $\mathcal{N}=1$ language, such that the flavour and colour structures essentially decouple. However, in the case of n -point next-to-extremal correlators there is more than one invariant contraction of the flavour indices in general. Therefore, picking just one particular $SU(3)$ representation for each operator is not general enough. Our strategy will be to use the full $SU(4)$ structure to constrain the possible Wick contractions. Then we show that each of these graphs vanishes for arbitrary $SU(3)$ representations.

The paper is organized as follows. In Section 2 we show that the simplest four-point next-to-extremal correlator, $\langle \text{Tr} X^4 \text{Tr} X^2 \text{Tr} X^2 \text{Tr} X^2 \rangle$, is not renormalized to order g^2 . In Section 3, the proof is extended to any four-point next-to-extremal correlator, and in Section 4, to any next-to-extremal correlator with n number of points. In Section 5 we discuss next-to-extremal four-point functions in AdS supergravity. Finally, Section 6 is devoted to conclusions, including a short discussion of instantonic corrections.

In terms of component fields, the Lagrangian of the $\mathcal{N}=4$ theory in Euclidean space reads:

$$\begin{aligned} \mathcal{L} = & \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} \bar{\lambda} \not{D} \lambda + \overline{D_\mu z^i} D_\mu z^i + \frac{1}{2} \bar{\psi}^i \not{D} \psi^i \\ & + i\sqrt{2} g f_{abc} (\bar{\lambda}_a \bar{z}_b^i L \psi_a^i - \bar{\psi}_a^i R z_b^i \lambda_c) - \frac{1}{\sqrt{2}} g f_{abc} \epsilon^{ijk} (\bar{\psi}_a^i L z_b^j \psi_c^k + \bar{\psi}_a^i R \bar{z}_b^j \psi_c^k) \\ & - \frac{1}{2} g^2 (f_{abc} \bar{z}_b^i z_c^i)^2 + \frac{1}{2} g^2 f_{abc} f_{ade} \epsilon^{ijk} \epsilon^{ilm} z_b^i z_c^k \bar{z}_d^l \bar{z}_e^m. \end{aligned} \quad (1)$$

L and R are chirality projectors. The three complex fields z^i ($i = 1, 2, 3$) are combinations of the six fundamental real scalars of $\mathcal{N}=4$ SYM, X^I ($I = 1, \dots, 6$):

$$z^i = \frac{1}{\sqrt{2}} (X^i + iX^{i+3}), \quad \bar{z}^i = \frac{1}{\sqrt{2}} (X^i - iX^{i+3}). \quad (2)$$

The fields X^I belong to the **6** of the $SU(4)$ R-symmetry group, whereas z^i and \bar{z}^i transform in the **3** and $\bar{\mathbf{3}}$, respectively, of its $SU(3)$ subgroup. All fields are in the adjoint representation of the $SU(N)$ gauge group. It will be useful to introduce dashed lines to represent the effective vertex obtained from the two channels for a gluon exchange plus the four scalar contact interaction, as indicated in Fig. 1. The oriented solid lines indicate propagators connecting a z and a \bar{z} field, with the arrow going from z to \bar{z} . On the other hand, we shall use solid

lines without arrows to represent propagators of two X fields. The unoriented lines are equivalent to the sum of two oriented lines with opposite orientations, since $X^I(x)X^I(y) = z^i(x)\bar{z}^i(y) + \bar{z}^i(x)z^i(y)$.

2 The correlator $\langle \text{Tr}X^4 \text{Tr}X^2 \text{Tr}X^2 \text{Tr}X^2 \rangle$

We consider first, for simplicity, the order g^2 corrections to the next-to-extremal correlation function $\langle \text{Tr}X^4(w)\text{Tr}X^2(x)\text{Tr}X^2(y)\text{Tr}X^2(z) \rangle$. The single-trace chiral primary operators $\text{Tr}X^k \equiv \text{Tr}X^{i_1}X^{i_2}\dots X^{i_k}$ are traceless symmetric tensor products of the fields X^i . So, the operators $\text{Tr}X^4$ and $\text{Tr}X^2$ are in the **105** (with Dynkin labels $[0, 4, 0]$) and the **20'** ($[0, 2, 0]$) of $SU(4)$, respectively. These operators can be written as

$$\begin{aligned} \text{Tr}X^k &= X_{a_1}^{i_1}\dots X_{a_k}^{i_k}\text{Tr}(T^{a_1}\dots T^{a_k}) \\ &= X_{a_1}^{<i_1}\dots X_{a_k}^{i_k>}\text{Str}(T^{a_1}\dots T^{a_k}), \end{aligned} \quad (3)$$

where $X^{<i_1}\dots X^{i_k>}$ denotes a (nonsymmetric) traceless tensor product, T^a are the hermitian generators of $SU(N)$ with normalization $\text{Tr}(T^a T^b) = \delta^{ab}/2$ and the symmetric trace is defined as

$$\text{Str}(T^{a_1}\dots T^{a_k}) = \sum_{\text{perms } \sigma} \frac{1}{k!} \text{Tr}(T^{a_{\sigma(1)}}\dots T^{a_{\sigma(k)}}). \quad (4)$$

Here and in the following we use the notation of Ref. [11]. The case of multi-trace chiral primary operators will be discussed in the conclusions.

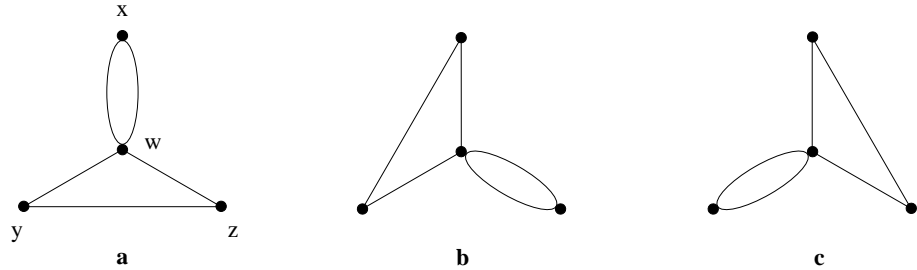


Figure 2: Feynman diagrams contributing to the correlator $\langle \text{Tr}X^4(w)\text{Tr}X^2(x)\text{Tr}X^2(y)\text{Tr}X^2(z) \rangle$ at the free-field level.

Disconnected diagrams do not contribute to this correlation function because they factorize into two connected subdiagrams, and one of them ($\langle \text{Tr}X^4(w)\text{Tr}X^2(x) \rangle$, for instance) cannot be a $SU(4)$ singlet. The connected Feynman diagrams contributing at the free-field level are depicted in Fig. 2. Note that diagrams with a scalar loop attached to a single point (a

“tadpole”) vanish due to the tracelessness of the operator $\text{Tr}X^4$. Diagram *a* gives the following free field correlation function:

$$\begin{aligned} & \langle \text{Tr}X^4(w) \text{Tr}X^2(x) \text{Tr}X^2(y) \text{Tr}X^2(z) \rangle \\ &= \mathcal{C}^{I_1 \dots I_4; J_1 J_2; K_1 K_2; L_1 L_2} G(w, x)^2 G(w, y) G(w, z) G(y, z) Q_{4,2,2,2}(N), \end{aligned} \quad (5)$$

where I_i , J_i , K_i and L_i are the flavour indices of each operator, \mathcal{C} is a tensor in flavour space, $G(x, y) = 1/(4\pi^2(x - y)^2)$ is the scalar propagator and

$$Q_{4,2,2,2} = \text{Str}(T^{a_1} \dots T^{a_4}) \text{Str}(T^{a_1} T^{a_2}) \text{Str}(T^{a_3} T^b) \text{Str}(T^{a_4} T^b) \quad (6)$$

contains all the colour structure. The space-time structure factorizes into a two-point function, $G(w, x)^2$, times a three-point function, $G(w, y)G(w, z)G(y, z)$. The contributions of diagrams *b* and *c* are obtained by the permutations $(x, J_i) \leftrightarrow (y, K_i)$ and $(x, J_i) \leftrightarrow (z, L_i)$, respectively. In the future we shall not consider explicitly the permuted diagrams, since they can be treated in exactly the same way.

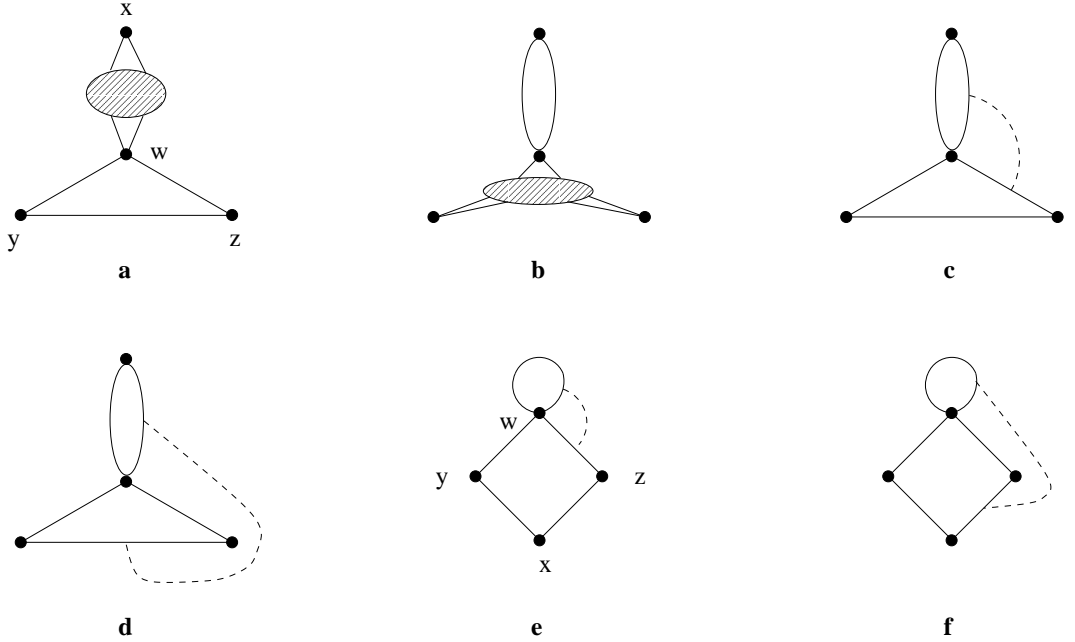


Figure 3: Feynman diagrams contributing to the correlator $\langle \text{Tr}X^4(w) \text{Tr}X^2(x) \text{Tr}X^2(y) \text{Tr}X^2(z) \rangle$ at order g^2 . We do not show explicitly diagrams that are obtained from these by permutations of the operators. In diagrams *c* and *d*, the dashed line can be connected to any of the two solid lines between x and w .

The diagrams contributing to this correlation function at order g^2 (up to permutations of the operators) are depicted in Fig. 3. We shall show that each of them vanishes separately. For diagrams (actually, groups of diagrams) *a* and *b*, this is due to known non-renormalization

theorems for 2- and 3-point functions, respectively. Let us consider diagram b , which renders the contribution

$$\begin{aligned} & \langle [X_{b_1}^{\{J_1\}} X_{b_2}^{\{J_2\}}](x) [X_{a_1}^{\{I_1\}} X_{a_2}^{\{I_2\}}](w) \rangle^{(0)} \langle [X_{a_3}^{\{I_3\}} X_{a_4}^{\{I_4\}}](w) [X_{c_1}^{\{K_1\}} X_{c_2}^{\{K_2\}}](y) [X_{d_1}^{\{L_1\}} X_{d_2}^{\{L_2\}}](z) \rangle^{(1)} \\ & \cdot \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \text{Tr}(T^{b_1} T^{b_2}) \text{Tr}(T^{c_1} T^{c_2}) \text{Tr}(T^{d_1} T^{d_2}), \end{aligned} \quad (7)$$

where the superindices (0) and (1) indicate free-field and first (g^2) order correlators, respectively. In (7), the two flavour indices J_1 and J_2 of $\text{Tr} X^2(x)$ are contracted with two of the indices of $\text{Tr} X^4(w)$, I_1, \dots, I_4 , within the free two-point function. This leaves a symmetric traceless tensor of rank 2. Therefore, as far as the flavour structure is concerned, the operator entering the three-point function at w is in the $\mathbf{20}'$ of $SU(4)$. Likewise, the contraction of the colour indices b_1 and b_2 with a_1 and a_2 within the two-point function, and the normalization $\text{Tr}(T^{b_1} T^{b_2}) = \delta^{b_1 b_2}/2$, imply that the colour indices of the two generators T^{a_1}, T^{a_2} are contracted to give the Casimir of their representation, leaving only T^{a_3} and T^{a_4} in that trace. Thus, the (order g^2) three-point function factor is

$$\langle \text{Tr} X^2(w) \text{Tr} X^2(y) \text{Tr} X^2(z) \rangle^{(1)}, \quad (8)$$

with three chiral primaries in the $\mathbf{20}'$. This expression must vanish, since it is known that the correlator $\langle \text{Tr}(X^2)(w) \text{Tr} X^2(y) \text{Tr} X^2(z) \rangle$ does not receive quantum corrections. Hence, diagram b vanishes. A similar argument holds for diagram a , which vanishes as well.

Any of the diagrams of Fig. 3 can be decomposed into a sum of “oriented” diagrams with arrows on the scalar lines. Different “orientations” correspond either to different $SU(3)$ representations in the $SU(4) \rightarrow SU(3) \times U(1)$ decomposition of the operators or to different Wick contractions of a given set of $SU(3)$ representations². The explicit colour and flavour structure of each diagram depends on the assignment of arrows to the scalar lines. However, all the terms in the possible oriented diagrams contain the structure $f_{abp} f_{cdp}$, where a, b, c and d are the colour indices of the four scalar lines involved in the gluon exchange or the contact interaction. In the case of gluon exchange and of the contact interaction coming from the D -term in the Lagrangian, the colour indices a and b (and c and d) are attached to one incoming and one outgoing scalar line, while for the F -term contact interaction, they are attached to two incoming or two outgoing lines. The essential fact for the present argument is that, for fixed colour indices in the relevant scalar propagators, there are only three possible colour combinations: $f_{abp} f_{cdp}$, $f_{acp} f_{bdp}$ and $f_{adp} f_{bcp}$. This structure is enough to prove that any oriented diagram corresponding to the diagrams c, d, e and f vanishes.

Let us consider diagram c . For any given orientation, its colour structure is given by:

$$\begin{aligned} & \text{Str}(T^{a_1} \dots T^{a_4}) \text{Str}(T^{a_1} T^b) \text{Str}(T^c T^d) \text{Str}(T^d T^{a_4}) \\ & (A f_{a_2 b p} f_{a_3 c p} + B f_{a_2 c p} f_{a_3 b p} + C f_{a_2 a_3 p} f_{b c p}) \\ & = \frac{1}{6} \text{Str}(T^{a_1} \dots T^{a_4}) \cdot (A f_{a_2 a_1 p} f_{a_3 a_4 p} + B f_{a_2 a_4 p} f_{a_3 a_1 p} + C f_{a_2 a_3 p} f_{a_1 a_4 p}) \\ & = 0, \end{aligned} \quad (9)$$

²The chiral primary operators are linear combinations of $SU(3)$ irreducible representations. For instance, if $I = i, J = j \in \{1, 2, 3\}$, $\text{Tr} X^{\{I\}} X^{\{J\}} = \frac{1}{2} \text{Tr}(z^{(i} z^{j)} + \bar{z}^{(i} \bar{z}^{j)} + z^{\{i} \bar{z}^{j\}} + \bar{z}^{\{i} z^{j\}})$.

where A , B and C are functions of the points w , x , y and z and of the flavour indices, and we have used $\text{Tr}(T^a T^b) = \delta^{ab}/2$. Expression (9) vanishes since, for each term, two colour indices of the totally symmetric $\text{Str}(T^{a_1} \dots T^{a_4})$ are contracted with two indices of an antisymmetric structure constant. The remaining diagrams, d , e and f , vanish for the same reason: they involve a contraction of symmetric and antisymmetric tensors as well.

Therefore, the correlator $\langle \text{Tr} X^4(w) \text{Tr} X^2(x) \text{Tr} X^2(y) \text{Tr} X^2(z) \rangle$ does not receive quantum corrections at order g^2 . For this correlation function, the proof is simplified by the fact that the trace of $T^a T^b$ gives a Kronecker delta. In the next section we shall see that the argument can be generalized to operators of higher dimension.

3 Next-to-extremal four-point correlators

In this section we consider the correlation function $\langle \text{Tr} X^{k_1}(w) \text{Tr} X^{k_2}(x) \text{Tr} X^{k_3}(y) \text{Tr} X^{k_4}(z) \rangle$, with $k_1 = k_2 + k_3 + k_4 - 2$ and $k_i \geq 2$. Each operator is in the $SU(4)$ irreducible representation $[0, k_i, 0]$. In order to show that its radiative corrections vanish at first order, we shall group—as we did in last section—all the diagrams whose space-time part factorizes into a free two-point subdiagram times a three-point subdiagram at order g^2 , or the other way round. Then, we shall invoke the non-renormalization theorems proved in [11] for two- and three-point functions to conclude that the sum of all these diagrams cancel. As we shall discuss, the proofs given in [11] apply also to the mentioned subdiagrams, which involve chiral primaries with insertions of extra $SU(N)$ generators in the trace of one of the operators. For the other diagrams, we shall see that they vanish due to their colour structure.

Let us consider first the free-field contributions. It is impossible to construct a $SU(4)$ invariant disconnected diagram, since (for i, j, l, m all different) $k_i = k_j$ and $k_l = k_m$ cannot hold at the same time. The next-to-extremality condition, together with tracelessness in flavour space, puts also severe restrictions on the possible connected Feynman graphs.

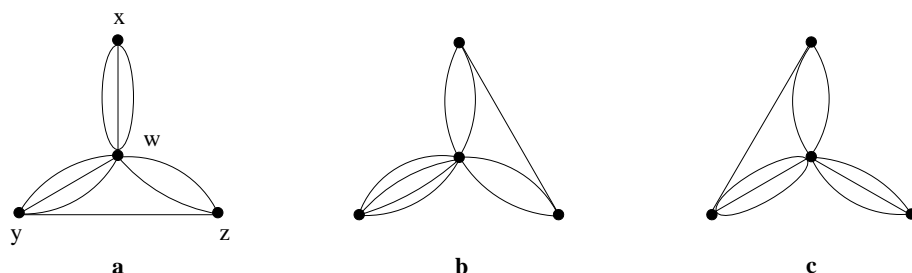


Figure 4: Feynman diagrams contributing to the correlator $\langle \text{Tr} X^{k_1}(w) \text{Tr} X^{k_2}(x) \text{Tr} X^{k_3}(y) \text{Tr} X^{k_4}(z) \rangle$ at the free-field level. Diagrams are drawn for $k_1 = 8$, $k_2 = 3$, $k_3 = 4$ and $k_4 = 3$. For arbitrary values of k_i , solid lines have to be added to or removed from the different “rainbows”.

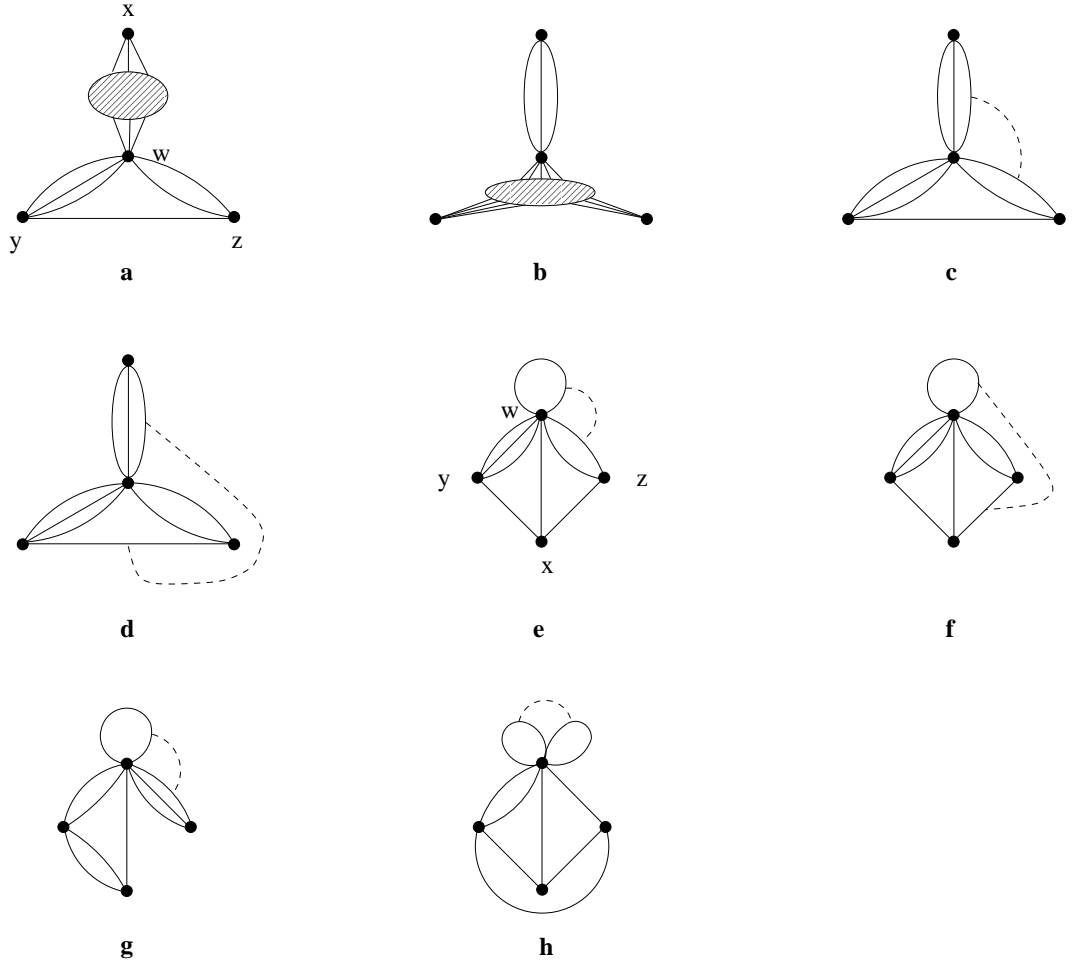


Figure 5: Feynman diagrams contributing to the correlator $\langle \text{Tr} X^{k_1}(w) \text{Tr} X^{k_2}(x) \text{Tr} X^{k_3}(y) \text{Tr} X^{k_4}(z) \rangle$ at order g^2 . As in Fig. 4, diagrams are drawn for $k_1 = 8$, $k_2 = 3$, $k_3 = 4$ and $k_4 = 3$. The dashed lines can be connected to any of the solid lines in each “rainbow”. Here and in the following, permutations are not shown explicitly. Diagram h is just one of the various diagrams that have two tadpoles connected by a dashed line.

The only diagrams contributing at the free-field level are depicted in Fig. 4. They all have the same structure, so we shall focus on the first one, which yields

$$\mathcal{C}^{I_1 \dots I_{k_1}; J_1 \dots J_{k_2}; K_1 \dots K_{k_3}; L_1 \dots L_{k_4}} G(w, x)^{k_2} G(w, y)^{k_3-1} G(w, z)^{k_4-1} G(y, z) Q_{k_1, k_2, k_3, k_4}(N), \quad (10)$$

where

$$Q_{k_1, k_2, k_3, k_4} = \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}(T^{a_1} \dots T^{a_{k_2}}) \cdot \text{Str}(T^{a_{k_2}+1} \dots T^{a_{k_2+k_3-1}} T^b) \text{Str}(T^{a_{k_2+k_3}} \dots T^{a_{k_1}} T^b). \quad (11)$$

We see that in the free-field approximation the space-time structure factorizes into a two-point function and a three-point function. The two-point function consists of a “rainbow” of scalar

lines connecting w and x , while the three-point function has two “rainbows” connecting w to y and z , and one additional scalar line between y and z .

At order g^2 , the possible kinds of diagrams are depicted in Fig. 5. In addition to the diagrams we show, there are others obtained by permutations of the operators at x , y , and z , and by attaching the dashed line to different scalar lines in the “rainbows”. It will be sufficient to study the set of diagrams in Fig. 6, for the remaining diagrams can be shown to vanish in exactly the same manner. As in the case of the correlator $\langle \text{Tr} X^4 \text{Tr} X^2 \text{Tr} X^2 \text{Tr} X^2 \rangle$, diagrams a and b only contain radiative corrections to a two-point and a three point function, respectively. In this case their vanishing follows from a simple generalization of the proof in [11] of the non-renormalization of general correlators of two or three chiral primaries.

Let us start with diagram a , which gives the contribution

$$\begin{aligned} & \langle [X_{b_1}^{\{J_1\}} \dots X_{b_{k_2}}^{\{J_{k_2}\}}](x) [X_{a_1}^{\{I_1\}} \dots X_{a_{k_2}}^{\{I_{k_2}\}}](w) \rangle^{(1)} \\ & \cdot \langle [X_{a_{k_2+1}}^{\{I_{k_2+1}\}} \dots X_{a_{k_1}}^{\{I_{k_1}\}}](w) [X_{c_1}^{\{K_1\}} \dots X_{c_{k_3}}^{\{K_{k_3}\}}](y) [X_{d_1}^{\{L_1\}} \dots X_{d_{k_4}}^{\{L_{k_4}\}}](z) \rangle^{(0)} \\ & \cdot \text{Tr}(T^{a_1} \dots T^{a_{k_1}}) \text{Tr}(T^{b_1} \dots T^{b_{k_2}}) \text{Tr}(T^{c_1} \dots T^{c_{k_3}}) \text{Tr}(T^{d_1} \dots T^{d_{k_4}}). \end{aligned} \quad (12)$$

The operator entering the two point function at w is a symmetric traceless tensor in flavour space, so it belongs to the $[0, k_2, 0]$ representation of $SU(4)$. Therefore we are lead to consider the two-point function of two chiral primaries of conformal dimension k_2 in the representation $[0, k_2, 0]$, but with $k_1 - k_2$ additional $SU(N)$ generators inserted in one of the traces:

$$\langle \text{Tr} X^{k_2}(x) \text{Tr}(X^{k_2}(w) T^{a_{k_2+1}} \dots T^{a_{k_1}}) \rangle. \quad (13)$$

The next step is to realize that the proof in [11] of the non-renormalization of the correlator $\langle \text{Tr} X^k \text{Tr} X^k \rangle$ applies also to the two-point correlator (13). Indeed, it is clear that self-energy corrections are not affected by the extra generators in the second trace. On the other hand, the pairs of structure constants appearing in gluon exchanges and quartic vertices can be converted into commutators within the *first* trace. The sum of all these diagrams can then be reduced to k_2 identical terms, as in Eqs. (2.5) and (2.6) of [11]. The key point is that there are not additional factors in the first trace that would give extra terms upon the application of the identity

$$\sum_{i=1}^n \text{Tr}(M_1 \dots [N, M_i] \dots M_n) = 0. \quad (14)$$

Note that the presence of extra generators inside both traces would prevent us from using this argument. Therefore, after adding all contributions one finds that the radiative corrections to the correlator in Eq. (13) are of the form of Eq.(2.7) in [11], and hence the non-renormalization theorem for $\langle \text{Tr} X^2 \text{Tr} X^2 \rangle$ implies that they vanish.

In the same manner, the non-renormalization theorems for three-point functions imply that diagram b vanishes. Here, we must consider the correlator

$$\langle \text{Tr}(T^{a_1} \dots T^{a_{k_2}} X^{k_1-k_2}(w)) \text{Tr} X^{k_3}(y) \text{Tr} X^{k_4}(z) \rangle. \quad (15)$$

Self-energy corrections are again trivial, and interactions relating scalar lines within one “rainbow” can be treated as in Eq. (4.7) of [11] if one chooses to put the commutators inside the second or third trace, which is always possible. Furthermore, one can easily see that the manipulations carried out in [11] for diagrams with interactions among two different “rainbows” can also be done in our case. Indeed, as long as only one of the traces contains extra T ’s, there is no obstruction to the use of Eq. (14). Therefore, the sum of all contributions is of the form of Eq. (4.9) of [11] and vanishes due to the non-renormalization theorems for $\langle \text{Tr} X^2 \text{Tr} X^2 \rangle$ and $\langle \text{Tr} X^2 \text{Tr} X^2 \text{Tr} X^2 \rangle$.

Let us study now the genuine four-point diagrams. Each of them is a linear combination of oriented diagrams. The general color structure of the interactions, which was discussed in last section, will be sufficient again to show that all possible oriented diagrams vanish. Consider diagram c of Fig. 5. For any given orientation, it gives a contribution of the following form:

$$\begin{aligned} & \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}(T^{a_1} \dots T^{a_{k_2-1}} T^b) \text{Str}(T^c T^{a_{k_2}+2} \dots T^{a_{k_2}+k_3-1} T^d) \\ & \cdot \text{Str}(T^d T^{a_{k_2}+k_3} \dots T^{a_{k_1}}) \left(A' f_{a_{k_2} b p} f_{a_{k_2}+1 c p} + B' f_{a_{k_2} c p} f_{a_{k_2}+1 b p} + C' f_{a_{k_2} a_{k_2}+1 p} f_{b c p} \right), \quad (16) \end{aligned}$$

where A' , B' and C' carry the space-time and flavour dependence. The term involving C' vanishes because $f_{a_{k_2} a_{k_2}+1 p}$ is contracted to a trace that is symmetric in $a_{k_2}, a_{k_2}+1$. We convert one of the structure constants in the terms involving A' (B') into a commutator inside the second trace, and then use Eq. (14) to obtain

$$\begin{aligned} & i \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}(T^c T^{a_{k_2}+2} \dots T^{a_{k_2}+k_3-1} T^d) \text{Str}(T^d T^{a_{k_2}+k_3} \dots T^{a_{k_1}}) \\ & \cdot \left\{ A' f_{a_{k_2}+1 c p} \text{Str}(T^{a_1} \dots T^{a_{k_2-1}} [T^{a_{k_2}}, T^p]) + B' f_{a_{k_2} c p} \text{Str}(T^{a_1} \dots T^{a_{k_2-1}} [T^{a_{k_2}+1}, T^p]) \right\} \\ & = -i \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}(T^c T^{a_{k_2}+2} \dots T^{a_{k_2}+k_3-1} T^d) \text{Str}(T^d T^{a_{k_2}+k_3} \dots T^{a_{k_1}}) \\ & \cdot \sum_{i=1}^{k_2-1} \left\{ A f_{a_{k_2}+1 c p} \text{Str}(T^{a_1} \dots [T^{a_{k_2}}, T^{a_i}] \dots T^{a_{k_2-1}} T^p) \right. \\ & \quad \left. + B f_{a_{k_2} c p} \text{Str}(T^{a_1} \dots [T^{a_{k_2}+1}, T^{a_i}] \dots T^{a_{k_2-1}} T^p) \right\}. \quad (17) \end{aligned}$$

The traces must not be symmetrized in the generators arising from the structure constants. Each term contains a commutator that is antisymmetric in a_{k_2}, a_i or in $a_{k_2}+1, a_i$ and is contracted to a trace symmetric in those indices. Therefore the entire diagram vanishes.

Diagram d gives the contribution

$$\begin{aligned} & \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}(T^{a_1} \dots T^{a_{k_2-1}} T^b) \text{Str}(T^{a_{k_2}+1} \dots T^{a_{k_2}+k_3-1} T^c) \\ & \cdot \text{Str}(T^d T^{a_{k_2}+k_3} \dots T^{a_{k_1}}) \left(A'' f_{a_{k_2} b p} f_{c d p} + B'' f_{a_{k_2} c p} f_{d b p} + C'' f_{a_{k_2} d p} f_{b c p} \right). \quad (18) \end{aligned}$$

For the terms involving A'' , B'' and C'' , the structure constant with the index a_{k_2} can be converted into a commutator inside the second, third and fourth trace, respectively. The identity (14) then leads to a sum of terms with commutators $[T^{a_{k_2}}, T^{a_i}]$ that cancel when

contracted with the first symmetric trace. Let us show this explicitly for the term involving B'' :

$$\begin{aligned}
& iB'' f_{dbp} \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}(T^{a_1} \dots T^{a_{k_2-1}} T^b) \\
& \quad \cdot \text{Str}(T^{a_{k_2+1}} \dots T^{a_{k_2+k_3-1}} [T^{a_{k_2}}, T^p]) \text{Str}(T^d T^{a_{k_2+k_3}} \dots T^{a_{k_1}}) \\
& = -iB f_{dbp} \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}(T^{a_1} \dots T^{a_{k_2-1}} T^b) \\
& \quad \cdot \sum_{i=k_2+1}^{k_2+k_3-1} \text{Str}(T^{a_{k_2+1}} \dots [T^{a_{k_2}}, T^{a_i}] \dots T^{a_{k_2+k_3-1}} T^p) \text{Str}(T^d T^{a_{k_2+k_3}} \dots T^{a_{k_1}}) \\
& = 0
\end{aligned} \tag{19}$$

It is interesting to note that this argument would not hold if there were another scalar line connecting the operators at y and z . Such a diagram is forbidden by next-to-extremality, but it can contribute to next-to-next-to-extremal correlators.

The remaining diagrams have a dashed line connected to a tadpole. All diagrams with tadpoles must have interactions involving all the tadpole lines, for otherwise they would vanish due to the tracelessness of the operators. Diagrams e , g and h of Fig. 6 vanish because all possible terms contain the factor

$$f_{a_i a_j p} \text{Str}(T^{a_1} \dots T^{a_{k_1}}) = 0, \tag{20}$$

where $i = 1, 2$, $j = 1, \dots, k_1$. Finally, diagram f has three possible terms, one proportional to the color structure in Eq. (20) and two to a factor of the form ($i = 1, 2$):

$$\begin{aligned}
& f_{a_i b p} \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}(T^b T^{a_{k_2+1}} \dots T^{a_{k_2+k_3-1}}) \\
& = i \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \text{Str}([T^{a_i}, T^p] T^{a_{k_2+1}} \dots T^{a_{k_2+k_3-1}}) \\
& = -i \text{Str}(T^{a_1} \dots T^{a_{k_1}}) \sum_{j=k_2+1}^{k_2+k_3-1} \text{Str}([T^p T^{a_{k_2+1}} \dots [T^{a_i}, T^{a_j}] \dots T^{a_{k_2+k_3-1}}) \\
& = 0.
\end{aligned} \tag{21}$$

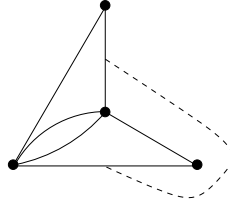


Figure 6: A Feynman diagram contributing to a next-to-next-to-extremal correlator at order g^2 .

Therefore, we conclude that the correlator $\langle \text{Tr} X^{k_1}(w) \text{Tr} X^{k_2}(x) \text{Tr} X^{k_3}(y) \text{Tr} X^{k_4}(z) \rangle$ does not receive quantum corrections at order g^2 . On the other hand, in the case of four-point correlators that are neither extremal nor next-to-extremal, non-vanishing Feynman diagrams (as the one illustrated in Fig. 6) contribute at order g^2 , and there is no reason for them to cancel.

4 General next-to-extremal correlators

The arguments given in last section can be directly extended to a general n -point function, $\langle \text{Tr} X^{k_1}(x_1) \text{Tr} X^{k_2}(x_2) \cdots \text{Tr} X^{k_n}(x_n) \rangle$, with $k_1 = \sum_{i=2}^n k_i - 2$ and $k_i \geq 2$. As in the case of four-point next-to-extremal correlators, disconnected diagrams are forbidden by $SU(4)$ symmetry. The reason is that the next-to-extremality condition implies that $k_1 > \sum_{i=1}^n k_i - k_l - k_m$ for any l, m , so that if two or more operators form a subdiagram disconnected from the rest of the diagram, it is impossible to build a $SU(4)$ invariant subdiagram with the remaining operators.

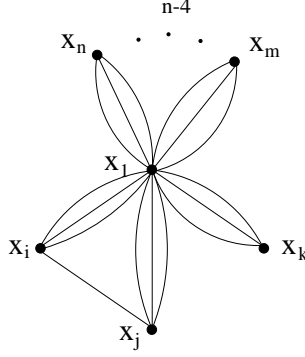


Figure 7: Feynman diagrams contributing to an n -point next-to-extremal correlator at the free-field level.

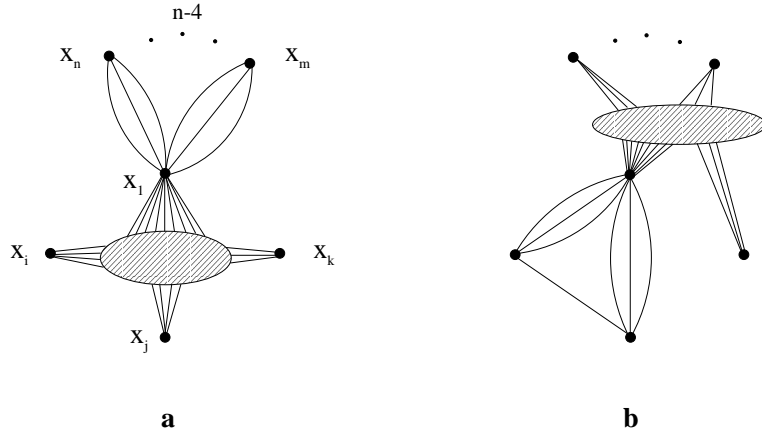


Figure 8: Feynman diagrams contributing to an n -point next-to-extremal correlator at order g^2 .

The free-field connected Feynman diagrams (Fig. 7) are analogous to the corresponding ones for four-point correlators: their space-time part factorizes into a product of $(n - 3)$ two-point functions times one three-point function. A similar structure is preserved at order g^2 : as is shown in Fig. 8, the space-time part of all diagrams factorizes either into a free extremal function of $(n - 4)$ points times a first-order four-point next-to-extremal function (diagram *a*) or into a free three-point next-to-extremal function times a first-order extremal function of $(n - 3)$ points (diagram *b*).

In diagram *a*, the operator that enters the four-point function at *w* is a traceless symmetric tensor of rank $k_i + k_j + k_l - 2$, where $i, j, l \in \{2, \dots, n\}$ label the other three operators involved in the four-point function. Hence, we are dealing with a next-to-extremal four-point correlator of the form

$$\langle \text{Tr}(T^{a_1} \dots T^{a_{k_1 - k_i - k_j - k_l + 2}} X^{k_i + k_j + k_l - 2}(x_1)) \text{Tr} X^{k_i}(x_i) \text{Tr} X^{k_j}(x_j) \text{Tr} X^{k_l}(x_l) \rangle \quad (22)$$

We have proved in Section 3 that next-to-extremal four-point correlators of chiral primaries are not renormalized, and we only need to make sure that the argument also holds in the presence of additional $SU(N)$ generators inside the first trace. This is indeed the case since the extra generators are included in the trace of the operator with highest dimension. Therefore, only one of the traces in the two- and three-point subdiagrams contributing to the correlator (22) contains extra generators, as is necessary for the non-renormalization of these subdiagrams. Furthermore, the diagrams that cannot be factorized into a two-point and a three-point part vanish as well since the property of the highest-dimension operator essential in the proof is that it is totally symmetric in the colour indices, and this is also true in the presence of additional generators. Therefore, next-to-extremal four-point correlators with extra group generators attached to the highest-dimension operator do not receive quantum corrections either.

Similarly, diagram *b* involves an extremal correlator of $n - 3$ chiral primaries, with extra $SU(N)$ generators attached to the highest-dimension operator:

$$\begin{aligned} & \langle \text{Tr}(T^{a_1} \dots T^{a_{k_i + k_j - 2}} X^{k_1 - k_i - k_j + 2}(x_1)) \text{Tr} X^{k_2}(x_2) \dots \text{Tr} X^{k_{i-1}}(x_{i-1}) \\ & \cdot \text{Tr} X^{k_{i+1}}(x_{i+1}) \dots \text{Tr} X^{k_{j-1}}(x_{j-1}) \text{Tr} X^{k_{j+1}}(x_{j+1}) \dots \text{Tr} X^{k_n}(x_n) \rangle. \end{aligned} \quad (23)$$

Here, *i* and *j* label the operators in the free-field three-point subdiagram. The extremal correlator in (23) can be shown to vanish using methods similar to the ones in Section 3. Alternatively, we may observe that the proof in [5] of the non-renormalization of extremal *n*-point correlators at order g^2 holds also in the presence of extra generators. The arguments in [5] are based on the space-time structure of the diagrams (which is colour independent) and on the symmetry of the colour indices in the highest-dimension operator, which is not affected by the extra generators.

Thus, we see that the combination of non-renormalization theorems for extremal *n*-point functions and for next-to-extremal four-point functions implies the vanishing of any next-to-extremal *n*-point function.

5 Next-to-extremal correlators within AdS supergravity

Further evidence for the non-renormalization of next-to-extremal correlation functions may be obtained from calculations in type IIB classical supergravity on $AdS_5 \times S_5$ which, according to the AdS/CFT correspondence, is dual to $\mathcal{N}=4$ SYM at strong coupling and large *N*.

The supergravity states corresponding to chiral primary operators of $\mathcal{N}=4$ SYM are certain scalar mixtures, s_k , of the trace of the graviton on S_5 and the five form field on S_5 . These one-particle states have the correct transformation properties under the superconformal group. They belong to the $[0, k, 0]$ representation of $SU(4)$ and have conformal dimension $\Delta = k$. In [12] the three-point amplitudes of these fields have been calculated and shown to be equal to their free-field values, in agreement with the field theory results [11, 13]. In [4] it has been argued that extremal n -point functions satisfy non-renormalization theorems as well, such that to all orders in perturbation theory they are given by a product of two-point functions whose coefficient does not receive quantum corrections either. Here we are interested in next-to-extremal functions of four chiral primary scalars, $s_{k_1}(x_1), \dots, s_{k_4}(x_4)$, with $k_1 = k_2 + k_3 + k_4 - 2$.

We use the methods for calculating correlation functions in AdS space developed in [14], [15], [16] and consider the Euclidean continuation of AdS_5 whose metric is given by

$$ds^2 = \frac{1}{z_0^2} (dz_0^2 + \sum_{i=1}^4 dz_i^2). \quad (24)$$

The scalar bulk to boundary propagator is given by [3], [14]

$$K_\Delta(z_0, \vec{z}, \vec{x}) = C_\Delta \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta, \quad C_\Delta = \frac{\Gamma(\Delta)}{\pi^2 \Gamma(\Delta - 2)}. \quad (25)$$

The bulk propagator $G_\Delta(u)$ depends on the chordal distance $u = 2z_0 y_0 / (z - y)^2$. Its explicit form is not needed here.

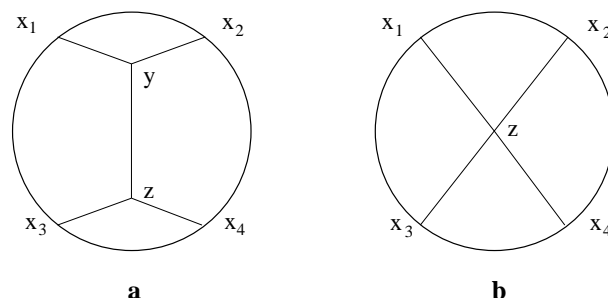


Figure 9: Witten diagrams contributing to next-to-extremal four-point functions at large N . To diagram *a* one should add the other two channels for particle exchange in the bulk.

We calculate the spatial dependence of the exchange diagrams contributing to next-to-extremal correlators by adapting an argument given in [4] for the extremal case. The two types of connected Witten diagrams contributing to four-point functions are depicted in Fig. 9. The quartic supergravity couplings required for the evaluation of the contact diagram are not yet known, so we concentrate on the exchange diagrams. Without loss of generality we consider the channel where $s_{k_1}(x_1)$ and $s_{k_2}(x_2)$ join at y to the intermediate field ϕ , which then splits at z into the other two fields, $s_{k_3}(x_3)$ and $s_{k_4}(x_4)$, as shown in Fig. 9. The exchanged field must be in a common $SU(4)$ representation of the tensor products $[0, k_1, 0] \times [0, k_2, 0]$ and

$[0, k_3, 0] \times [0, k_4, 0]$. It is easy to see using Young tableaux that the only common representations when $k_1 = k_2 + k_3 + k_4 - 2$ are $[0, k_3 + k_4 - 2, 0]$, $[0, k_3 + k_4, 0]$ and $[1, k_3 + k_4 - 2, 1]$. The exchanged field must be in one of these representations, and it can be either a primary field or a $SU(2, 2|4)$ descendent.

For scalar primary exchange there are two possibilities: $[0, k_3 + k_4 - 2, 0]$ and $[0, k_3 + k_4, 0]$. The exchanged field has dimension $k_3 + k_4 - 2$ or $k_3 + k_4$, respectively. In the first case, the cubic vertex at y is extremal and the vertex at z is next-to-extremal, while in the second case the vertex at y is next-to-extremal and the vertex at z is extremal. It is known from supergravity [12, 17] that the cubic couplings, denoted by $\mathcal{G}(k_1, k_2, k)$, vanish at extremality while they are finite in the subextremal case. On the other hand, divergent integrals appear in extremal correlators. We shall regularize them by analytic continuation in the conformal dimensions [18, 4]. It turns out that the zeros in the couplings combine with the poles in the integrals to give a finite result.

The exchange of the primary $s_{k_3+k_4-2}$ leads to the contribution

$$I_1 = \mathcal{G}(k_1, k_2, k_3 + k_4 - 2) \mathcal{G}(k_3, k_4, k_3 + k_4 - 2) \cdot \iint \frac{d^5 y}{y_0^5} \frac{d^5 z}{z_0^5} K_{\Delta_1}(x_1, y) K_{\Delta_2}(x_2, y) G_{\Delta_3+\Delta_4-2}(y, z) K_{\Delta_3}(x_3, z) K_{\Delta_4}(x_4, z). \quad (26)$$

The evaluation of this integral is based on realizing that the y -integrand, which leads to a pole since the vertex at y is extremal, is dominated by the contribution when $y \rightarrow x_1$. For $y \rightarrow x_1$ we may approximate the bulk propagator by [15, 19]

$$G_{\Delta} \longrightarrow \frac{1}{2\Delta - d} y_0^{\Delta} K_{\Delta}(\vec{x}_1, z), \quad (27)$$

such that the two integrals decouple and we have

$$I_1 = I_y \cdot I_z, \quad (28)$$

where

$$I_y = \mathcal{G}(k_1, k_2, k_3 + k_4 - 2) \frac{1}{x_{12}^{2\Delta_2}} \int \frac{d^5 y}{y_0^5} \frac{y_0^{\Delta_1+\Delta_2+\Delta_3+\Delta_4-2}}{(y_0^2 + (\vec{y} - \vec{x}_1)^2)^{\Delta_1}}, \quad (29)$$

$$I_z = \mathcal{G}(k_3, k_4, k_3 + k_4 - 2) \int \frac{d^5 z}{z_0^5} K_{\Delta_3+\Delta_4-2}(x_1, z) K_{\Delta_3}(x_3, z) K_{\Delta_4}(x_4, z). \quad (30)$$

I_z is a next-to-extremal finite three point function which may be evaluated using the methods of [14] to give

$$I_z \sim \frac{1}{x_{34}^2 x_{13}^{2(\Delta_3-1)} x_{14}^{2(\Delta_4-1)}}. \quad (31)$$

For the evaluation of I_y we use analytic continuation such that $\delta \equiv k_2 + k_3 + k_4 - 2 - k_1 \geq 0$, which implies $\mathcal{G}(k_1, k_2, k_3 + k_4 - 2 + \delta) \propto \delta$. Then, by translating the integration variable

$\vec{y} \rightarrow \vec{y} + \vec{x}_1$, and by rescaling $\vec{y} = y_0 \vec{w}$, we obtain

$$I_y = \delta \frac{1}{x_{12}^{2\Delta_2}} \int \frac{dy_0}{y_0^5} y_0^\delta \int \frac{d^4 w}{(1 + |\vec{w}|^2)^{\Delta_1}}. \quad (32)$$

The y_0 -integral gives a pole in δ which is exactly cancelled by the factor δ arising from the coupling. Thus the final result is of the form

$$I_1 \sim \frac{1}{x_{21}^{2\Delta_2} x_{34}^2 x_{13}^{2(\Delta_3-1)} x_{14}^{2(\Delta_4-1)}}, \quad (33)$$

which agrees with the free-field approximation.

The contribution from the exchange of the primary $s_{k_3+k_4}$ is

$$I_2 = \mathcal{G}(k_1, k_2, k_3 + k_4) \mathcal{G}(k_3, k_4, k_3 + k_4) \cdot \iint \frac{d^5 y}{y_0^5} \frac{d^5 z}{z_0^5} K_{\Delta_1}(x_1, y) K_{\Delta_2}(x_2, y) G_{\Delta_3+\Delta_4}(y, z) K_{\Delta_3}(x_3, z) K_{\Delta_4}(x_4, z). \quad (34)$$

This vanishes since the integral is finite and $\mathcal{G}(k_3, k_4, k_3 + k_4) = 0$. To see that the integral is finite we evaluate the y -integral by applying the method of [16] to the next-to-extremal vertex at y which yields

$$I_2 = \mathcal{G}(k_1, k_2, k_3 + k_4) \mathcal{G}(k_3, k_4, k_3 + k_4) \cdot \sum_{k=1}^{\Delta_2-1} a_k |x_{12}|^{-2\Delta_2+2k} \int \frac{d^5 z}{z_0^5} K_{\Delta_1-\Delta_2+k}(x_1, z) K_k(x_2, z) K_{\Delta_3}(x_3, z) K_{\Delta_4}(x_4, z), \quad (35)$$

with finite coefficients a_k . All integrals in the sum are finite as well.

The descendent exchange graphs are more involved, and we do not study them in full detail here, but give an argument from which we expect that descendent exchange does not contribute to next-to-extremal correlators. The superconformal descendent field ϕ can be a scalar or a tensor and must be in any of the three allowed representations: $[0, k_3 - k_4 - 2, 0]$, $[0, k_3 - k_4, 0]$ or $[1, k_3 - k_4, 1]$. Hence, it has a conformal dimension $\Delta \geq \Delta_3 + \Delta_4$ and is a descendent of a chiral primary in the $[0, k, 0]$ representation, with $k \geq k_3 + k_4$. The coupling constants for the cubic vertices $\phi s_{k_1} s_{k_2}$ and $\phi s_{k_3} s_{k_4}$ [17] are related to primary vertices by supersymmetry. The vertex $s_k s_{k_3} s_{k_4}$ is forbidden by $SU(4)$ symmetry if $k > k_3 + k_4$. Thus the only possibility is $k = k_3 + k_4$, which corresponds to an extremal vertex with a vanishing coupling constant. Therefore, the corresponding coupling for the vertex $\phi s_{k_3} s_{k_4}$ must also be zero. On the other hand, the vertex $s_k s_{k_1} s_{k_2}$ is next-to-extremal for $k = k_3 + k_4$. Since the conformal dimension of the field ϕ satisfies the relation $\Delta \geq \Delta_3 + \Delta_4$, these diagrams involve integrals similar to I_2 in Eq. (34). These integrals must be finite independently of the spin of the intermediate field, as the short distance behaviour of the bulk-to-bulk propagator is universal. We conclude that diagrams with exchange of descendents do not contribute to next-to-extremal functions.

Finally, for a complete argument showing that the AdS calculation yields the free-field result, it is necessary to calculate the contribution from the contact diagram. This is a next-to-extremal four-point scalar function, which is known to be finite. Since in this case the entire integration region contributes, and not just a small region surrounding one of the external points, logarithms are expected to appear and there seems to be no reason for these logarithms to cancel such as to give a product of two-point functions. Therefore, the quartic couplings of the primary fields, $\mathcal{G}(k_2 + k_3 + k_4 - 2, k_2, k_3, k_4)$, has to vanish if a free-field form is to be obtained. In the case corresponding to the correlator considered in Section 2, $\langle \text{Tr} X^4 \text{Tr} X^2 \text{Tr} X^2 \text{Tr} X^2 \rangle$, we can see the coupling constant $\mathcal{G}(4, 2, 2, 2)$ is indeed zero by the following argument. The field s_2 , corresponding to $\text{Tr} X^2$, is in the same multiplet as the graviton (in the lowest Kaluza-Klein level), whereas s_4 , corresponding to $\text{Tr} X^4$, is in a higher Kaluza-Klein level. On the other hand, it is believed that Type IIB supergravity on $AdS_5 \times S_5$ can be consistently truncated to include only the multiplet of the graviton, *i.e.*, the field content of the gauged $\mathcal{N}=8$, five-dimensional supergravity. Consistency requires that the equations of motion of the untruncated theory do not contain terms that are linear in higher Kaluza-Klein modes. Hence it implies, in particular, that $\mathcal{G}(4, 2, 2, 2)$ vanishes. More generally, the results in [6] and our argument for the exchange diagrams suggest that all the next-to-extremal quartic couplings of chiral primaries are zero.

6 Conclusion

We have proved that next-to-extremal correlation functions of single-trace chiral primary operators do not receive quantum corrections at order g^2 in perturbation theory. This result supports the (non-perturbative) results in [6] and generalizes them in the sense that it applies to correlators of any number of points.

Although we have dealt with single-trace operators for simplicity, next-to-extremal correlators of multi-trace chiral primaries have an analogous non-renormalization property. This can easily be shown using the non-renormalization theorems for two- and three-point functions of multi-trace primary operators proven in [20], and by noting that the relevant property for the vanishing of the diagrams that do not factorize into a two-point and a three-point function is that each operator has to be totally symmetric in its colour indices. Multi-trace operators have this property as well due to the general fact that primaries are symmetric tensors. Moreover, the $SU(4)$ structure is also the same as for single-trace operators.

We have studied next-to-extremal correlation functions in AdS supergravity as well. As in our field theory results, we have found that the exchange diagrams reduce to a product of two-point functions. Note that for I_1 in (26) this occurs in a way that resembles the field-theoretical calculation of diagrams *a* and *b* in Figs. 3 and 5 in Sections 2 and 3: the contribution (26) factorizes into a two-point and a three-point function, which both have a free-field form. On the other hand, I_2 in (34) corresponds to a contribution which does not

factorize into a two- and a three-point function, and the fact that it vanishes independently of its detailed space-time structure seems related to the fact that the non-factorizing field theory diagrams discussed in Sections 2 and 3 are zero as well. Of course, the AdS calculation is conjectured to describe the strong coupling regime, to which perturbation theory does not apply. Nevertheless, we see that the symmetries of the theory and the condition of next-to-extremality enter the calculations in a similar way both at weak and at strong coupling, such as to lead to agreeing results in both cases.

Very recently, the quartic couplings necessary for a detailed evaluation of the contact diagrams within AdS supergravity have been calculated in [10]. At least in the simplest case it is easy to see that there is agreement with the discussion here: The authors of [10] find that the quartic coupling involving three s_2 and a chiral primary in a higher Kaluza-Klein level vanishes, which agrees with consistent truncation of type IIB supergravity in $AdS_5 \times S_5$. Therefore our AdS calculation implies that the correlator $\langle \text{Tr} X^4 \text{Tr} X^2 \text{Tr} X^2 \text{Tr} X^2 \rangle$ has a free-field form for large g and large N . On the other hand, our perturbative result is valid for any N and thus supports the idea of consistent truncation at the quantum level. Furthermore, according to [6] the non-renormalization of next-to-extremal four-point functions is a non-perturbative effect. Hence we expect all the next-to-extremal quartic couplings in AdS supergravity to vanish, not just the simplest one discussed above. It would be interesting to see whether this agrees with the results in [10]. This check requires making field redefinitions as the ones carried out in [10] for the extremal case, and will be considered in future work. Moreover, our results suggest as a stronger conjecture that next-to-extremal couplings of n fields have to vanish as well.

The consideration of instanton configurations provides further non-perturbative evidence for the non-renormalization of next-to-extremal correlators. In [5], an instanton calculation has been performed for extremal correlators. The non-perturbative non-renormalization of extremal correlators was checked by computing the contribution from any instanton sector at leading order in the semiclassical expansion. These contributions were proven to vanish using both the systematics of gaugino zero-modes in the multi-instanton background [21] and the fact that any extremal correlator of irreducible representations in the $SU(3) \times U(1)$ decomposition of $SU(4)$ is related by $SU(4)$ transformations to a correlator of the form $\langle z^{k_1}(x_1) \bar{z}^{k_2}(x_2) \cdots \bar{z}^{k_n}(x_n) \rangle$. In the next-to-extremal case considered here there are $(n-1)(n-2)/2$ invariant contractions of the flavour indices. All the possible invariants are generated by the set of correlators in which the first operator is in a maximal weight representation: $\langle z^{k_1}(x_1) (\bar{z}^{k_2-1} z)(x_2) \bar{z}^{k_3}(x_3) \cdots \bar{z}^{k_n}(x_n) \rangle, \dots, \langle z^{k_1}(x_1) \bar{z}^{k_2}(x_2) \cdots \bar{z}^{k_{n-1}}(x_{n-1}) (\bar{z}^{k_n-1} z)(x_n) \rangle$, where $\bar{z}^{k_i-1} z$ denotes a traceless symmetric tensor of $SU(3)$ built out of $k_i - 1$ fields \bar{z} and one field z . In particular we may study correlation functions for maximal weight components in each tensor, such as $\langle (z^1)^{k_1}(x_1) ((\bar{z}^1)^{k_2-2} \bar{z} z)(x_2) (\bar{z}^1)^{k_3}(x_3) \cdots (\bar{z}^1)^{k_n}(x_n) \rangle$. The multi-instanton contribution to these correlators can be proven to vanish using an argument analogous to the one used in [5]: In $\mathcal{N}=4$ SYM a multi-instanton configuration has sixteen exact gaugino zero-modes ζ_α^A , corresponding to the eight supersymmetry and eight superconformal transformations broken by the instanton solution [22]. Solving the equations of motion of the scalar

fields in the instanton background, one finds (to order g^2) that the scalars are bilinear in the zero-modes and that z^1 (\bar{z}^1) contains zero-modes of flavour 0 and 1 (2 and 3) only, where 0 refers to the spinor in the vector multiplet. On the other hand, the measure of integration for the multi-instanton coordinates contains a factor $d^{16}\zeta$. In order to get a non-vanishing contribution, the first operator in the correlation function must absorb at least six of the eight zero-modes of type 0 and 1. Hence, this operator must contain a factor of $(\zeta)^3$. Since ζ is a two-component Grassmann variable, $(\zeta)^m = 0$ for any $m \geq 3$ and this factor (and the entire correlator) vanishes. This leads us to conclude that the next-to-extremal correlators do not receive corrections from any multi-instanton sector at leading order in the semiclassical expansion.

All these checks support the conjecture in [6] that next-to-extremal correlation functions of an arbitrary number of chiral primary operators are not renormalized.

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